↳ Mathematical definitions of R's basic properties on A

Mathematical definitions of the basic properties of the relation R on the set A:

Let R be a relation on A, then

(1) R is ***reflexive*** on A ***if and only if*** $I_A \subseteq R$

R at least contains all reflexive pairs (x,x)

(2) R is ***irreflexive*** on A ***if and only if*** $R \cap I_A = \varnothing$

R does not contain any (x,x)

(3) R is ***symmetric*** on A ***if and only if*** $R = R^{-1}$

(4) R is ***antisymmetric*** on A ***if and only if*** $R \cap R^{-1} \subseteq I_A$

R and its inverse relation contains only reflexive pairs ⟨x,x⟩

(5) R is ***transitive*** on A ***if and only if*** $R \circ R \subseteq R$

- To prove that R is reflexive on A:
  - **Proof Pattern:**

    For any x,

    $x \in A \Rightarrow \dots\dots\dots \quad \dots\dots\dots\dots\dots \quad \dots\dots\dots \Rightarrow \langle x, x\rangle \in R$

    Assume reasoning process Conclusion

e.g. **Example:** Prove that if $I_A \subseteq R$, then R is reflexive on A.

Proof: **For any x,**

$$x \in A \Rightarrow \langle x, x\rangle \in I_A \Rightarrow \langle x, x\rangle \in R$$

Therefore, R is reflexive on A.

↳ Proof of symmetric of R on A

- To prove that R is symmetric on A:
  - **Proof Pattern:**

    For any $<x, y>$

    $<x, y> \in R \Rightarrow \ldots\ldots \quad \ldots\ldots \ \ldots\ldots \quad \ldots\ldots \Rightarrow <y, x> \in R$

    Assumption Reasoning process Conclusion

e.g. **Example: Prove that if $R=R^{-1}$**, then R is symmetric on A.

Proof: Let $<x, y>$

$$<x, y> \in R \Rightarrow <y, x> \in R^{-1} \Rightarrow <y, x> \in R$$

Therefore, R is symmetric on A.

↳ Proof of antisymmetric of R on A

- To prove that R is antisymmetric on A:
  - **Proof Pattern:**

    For any $<x, y>$

    $<x, y>\in R \wedge <y, x>\in R \Rightarrow \ldots\ldots \quad \ldots\ldots \ \ldots\ldots \quad \ldots\ldots. \Rightarrow x=y$

    Assumption Reasoning process Conclusion

e.g. **Example: Prove that if** $R\cap R^{-1}\subseteq I_A$, ***then R is*** antisymmetric on A.

Proof: **For any $<x, y>$**

$$<x, y>\in R \wedge <y, x>\in R \Rightarrow <x, y>\in R \wedge <x, y>\in R^{-1}$$

$$\Rightarrow <x, y>\in R\cap R^{-1} \Rightarrow <x, y>\in I_A \Rightarrow x=y$$

Therefore, R is antisymmetric on A.

- To prove that R is transitive on *A*:
  - **Proof Pattern:**

    For any $<x, y>$, $<y, z>$

    $$<x, y>\in R\wedge<y, z>\in R \Rightarrow \ldots\ldots\ldots\ldots \Rightarrow <x, z>\in R$$

    Assumption Reasoning process Conclusion

e.g. **Example 7**: Prove that if $R\circ R\subseteq R$, **then** R is transitive on A.

Proof: **Let $<x, y>$, $<y, z>$**

$$<x, y>\in R \wedge<y, z>\in R \Rightarrow <x, z>\in R\circ R \Rightarrow <x, z>\in R$$

Therefore, R is transitive on A.

↳ Comparison Table of Properties of Relation R

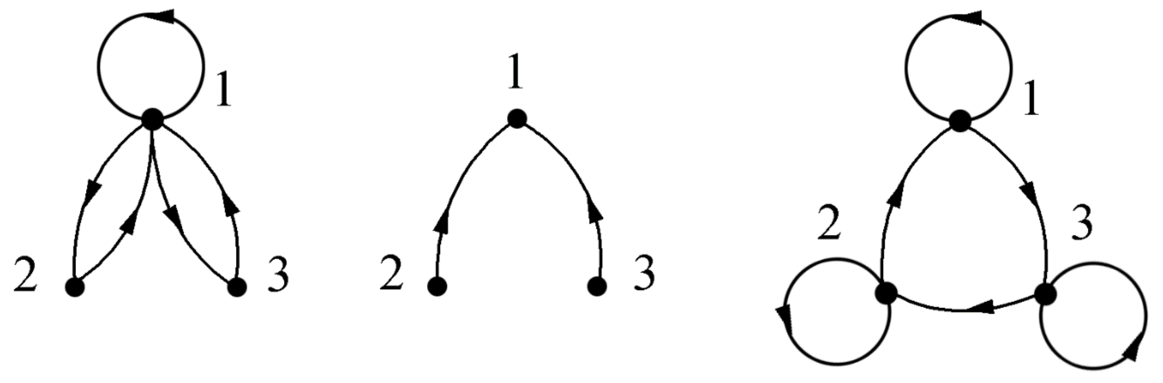
Relation Properties	Expression	Definition	Relation Matrix	Relation Diagram
Reflexivity	$I_A \subseteq R$	$\forall x \in A, \exists \langle x,x \rangle \in R$	Main diagonal elements are 1	Every vertex has a loop
Irreflexivity	$R \cap I_A = \varnothing$	$\forall x \in A, \exists \langle x,x \rangle \notin R$	Main diagonal elements are 0	No loops at any vertex
Symmetry	$R = R^{-1}$	If $\langle x,y \rangle \in R$, then $\langle y,x \rangle \in R$	The matrix is a symmetric matrix	If there is an edge between two vertices, it must be a directed edge (no undirected edge)

↳ Comparison Table of Properties of Relation *R* (cont.)

Relation Properties	Expression	Definition	Relation Matrix	Relation Diagram
Antisymmetry	$R \cap R^{-1} \subseteq I_A$	If Expression $<x,y> \in R$ and $x \neq y$, then $<y,x> \notin R$	If $r_{ij}=1$, and $i \neq j$, Then $r_{ji}=0$	If there is an edge between two points, it must be a directed edge (no bidirectional edges)
Transitivity	$R \circ R \subseteq R$	If $<x,y> \in R$ and $<y,z> \in R$, then $<x,z> \in R$	If $M_{ij}^2=1$, $M_{ij}=1$	If there is an edge from vertex x_i to x_j, and an edge from x_j to x_k, then there is also an edge from x_i to x_k.

↳ Determine the Relation Based on the Relation Diagram (e.g.)

e.g. **Example 8:** Determine the properties of the relationship in the figure below and explain the reasoning.



(a) (b) (c)

(a) Neither reflexive nor antireflexive; symmetric, not antisymmetric; not transitive.

(b) Antireflexive, not reflexive; antisymmetric, not symmetric; transitive.

(c) Reflexive, not antireflexive; antisymmetric, not symmetric; not transitive.

↳ Relation Between Operations and Properties

	Reflexivity	Irreflexivity	Symmetry	Antisymmetry	Transitivity
R_1^{-1}	√	√	√	√	√
$R_1 \cap R_2$	√	√	√	√	√
$R_1 \cup R_2$	√	√	√	×	×
$R_1 - R_2$	×	√	√	√	×
$R_1 \circ R_2$	√	×	×	×	×

- 4.3.1 Definition and Determination of Relation Properties
  - Reflexivity and Irreflexivity
  - Symmetry and Antisymmetry
  - Transitivity
- 4.3.2 Closure of Relations
  - Definition of Closure
  - Closure Calculation
  - Warshall's Algorithm

Definition 4.17: $r(R)$, $s(R)$ and $t(R)$

Let R be a relation on a non-empty set A. The **reflexive (symmetric or transitive) closure** of R is a relation R' *on* A, *such that* R' satisfies the following conditions:

- R' is reflexive (symmetric or transitive).
- $R \subseteq R'$
- For any reflexive (symmetric or transitive) relation R'' **on A that contains** R**, we have** $R' \subseteq R''$ **.**

The reflexive closure of R is usually denoted by $r(R)$, the symmetric closure by $s(R)$, and the transitive closure by $t(R)$.

↳ Construction of the Transitive Closure of Relation *R*

- For a relation R on a non-empty set A, the reflexive closure $r(R)$, symmetric closure $s(R)$, and transitive closure $t(R)$ can be constructed.
- The ***reflexive closure R′*** of R is a relation obtained by adding all necessary pairs to ensure reflexivity, and it is the smallest superset. It can be defined as: $R'=R\cup\{(a,a)\mid a\in A\}$
- The ***symmetric closure R′*** of R is a relation obtained by adding all necessary pairs to ensure symmetry, and it is the smallest superset. It can be defined as: $R'=R\cup\{(b,a)\mid(a,b)\in R\}$

↳ Construction of the Transitive Closure of Relation *R*

- The ***transitive closure*** R' of R is a relation obtained by adding all necessary pairs to ensure transitivity, and it is the smallest superset. It can be defined as:

For each pair of elements a,c∈A, if there exist one or more elements $b_1,b_2,\ldots,b_n$ such that $(a,b_1),(b_1,b_2),\ldots,(b_{n-1},b_n),(b_n,c)$ are all in R , then (a,c) should be in R'.

- **Theorem 4.7:** Closure Theorem of Relations.

  Let R be a relation on A, then we have:

  (1) $r(R)=R\cup R^0$

  (2) $s(R)=R\cup R^{-1}$

  (3) $t(R)=R\cup R^2\cup R^3\cup\ldots$

Explanation:

- For a finite set A (where $|A|=n$) , the union in (3) will have at most R^n.
- *If* R is reflexive, then $r(R)=R$; If R is symmetric, then $s(R)=R$; If R is transitive, then $t(R)=R$.

- **Proof of Theorem 4.7** (Proving (1)).
  - Proof of **(1)** $r(R)=R\cup R^0$, It is sufficient to show that $R\cup R^0$ satisfies the closure definition.
  - Proof that $R\cup R^0$ is a reflexive relation

    Since $R\cup R^0$ contains R , **and by** $I_A\subseteq R\cup R^0$, we can conclude that $R\cup R^0$ is reflexive on A.
  - Proof that $R\cup R^0$ is the smallest reflexive relation containing R .

    We need to show that no reflexive relation smaller than $R\cup R^0$ exists that contains R.

    Assume R' is a reflexive relation that contains R ***and*** and is smaller than $R\cup R^0$ $I_A\subseteq R'$, $R\subseteq R'$, Therefore, we have $R\cup R^0=I_A\cup R\subseteq R'$, which contradicts the assumption that R' is smaller than $R\cup R^0$.